Variation of the Eigenvalues of a Special Class of Hermitian Matrices upon Variation of Some of Its Elements

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ABSTRACT

For a Hermitian $n \times n$ matrix of the form

$$H = \begin{bmatrix} P & \rho Q \\ \bar{\rho} Q^* & R \end{bmatrix}$$

of which all the eigenvalues of the $s \times s$ submatrix P are greater than all the eigenvalues of the square $t \times t$ submatrix R it is proved that the s greater eigenvalues of H are increasing and the remaining t eigenvalues of H are decreasing functions of the absolute value of the complex variable ρ.

INTRODUCTION

Although it has been proved [2] that the eigenvalues of a matrix are continuous functions of the elements of the matrix, it is difficult to analyze the changing behavior of the eigenvalues in detail when certain elements of the matrix are modified in some prescribed way. A well-known study in this field is that of Perron and Frobenius [3], who have proved for a nonnegative irreducible matrix that the eigenvalue of greatest absolute magnitude, which is simple, real, and positive, increases when any matrix element increases.

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In this paper it will first be proved for Hermitian matrices of the form

$$H = \begin{bmatrix} P & \rho Q \\ \bar{\rho} Q^* & R \end{bmatrix},$$

where all the eigenvalues of P are greater than all the eigenvalues of R, that the eigenvalues of H are monotonous functions of the absolute value of the complex variable ρ. Two specific applications of this theory will then be discussed.

1. THEORY

A Hermitian matrix H is a square complex matrix of the form $H = A + iB$, where A is real and symmetric and B is real and skew symmetric. Hence we have for the transposes A^T and B^T the equations

$$A^T = A, \qquad B^T = -B. \tag{1.1}$$

From these relations it directly follows that, if H^* is the conjugate transpose of H, then $H^* = H$. Since, for any vector x, $(x^*Hx)^* = x^*H^*x = x^*Hx$, it follows that the eigenvalues of H are real. Further, we may assume, without loss of generality, that the eigenvalues of Hermitian matrices are arranged in nonincreasing order. Hence, if a Hermitian $n \times n$ matrix H has the eigenvalues λ_k $(1 \leqslant k \leqslant n)$, then

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n. \tag{1.2}$$

If such a matrix is partitioned in the form

$$H = \begin{bmatrix} P & Q \\ S & R \end{bmatrix}, \tag{1.3}$$

where P, Q, S, and R are $s \times s$, $s \times t$, $t \times s$, and $t \times t$ submatrices, respectively, then $P^* = P$, $Q^* = S$, and $R^* = R$; and from this it follows that the eigenvalues of P and R are also real.

If the eigenvalues of P and R lie in two separate intervals along the real axis, then certain relations exist between the eigenvalues of P and R and the eigenvalues of H. This will be enunciated in the following theorems.

SEPARATION THEOREM. *If the Hermitian matrix H can be partitioned in the form*

$$H = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \tag{1.4}$$

such that all the eigenvalues $\lambda_i^{(P)}$ $(1 \leqslant i \leqslant s)$ *of the square* $s \times s$ *submatrix* P *are greater than all the eigenvalues* $\lambda_j^{(R)}$ $(1 \leqslant j \leqslant t)$ *of the square* $t \times t$ *submatrix* R, *then no eigenvalue of* H *lies in the real interval* G, *defined by*

$$G = \{x | \max_j \lambda_j^{(R)} < x < \min_i \lambda_i^{(P)}\}.$$

Proof. Let λ be any eigenvalue of H, and v the associated eigenvector. Then

$$(H - \lambda I_n)v = 0, \tag{1.5}$$

I_n being the unit matrix of order n. In accordance with the partitioning (1.4) of H we can partition v thus:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1.6}$$

and hence obtain from (1.5) the equations

$$\begin{aligned} (P - \lambda I_s)v_1 + Qv_2 &= 0, \\ Q^*v_1 + (R - \lambda I_t)v_2 &= 0. \end{aligned} \tag{1.7}$$

From these equations follows immediately

$$v_1^*(P - \lambda I_s)v_1 + v_2^*(\lambda I_t - R)v_2 = 0. \tag{1.8}$$

If $\lambda \in G$, then $P - \lambda I_s$ and $\lambda I_t - R$ would be positive definite. Since $v \neq 0$, (1.8) would be impossible. Hence the assumption $\lambda \in G$ is not true, and as λ was any eigenvalue of H the proof is complete.

Example.

$$H = \begin{bmatrix} 2 & 3 & 2 \\ 3 & -2 & 0 \\ 2 & 0 & -3 \end{bmatrix}. \tag{1.9}$$

Taking $P = 2$ and

$$R = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

we find that no eigenvalue of H lies in the open interval $(-2, 2)$.

Note. The theorem does not apply to non-Hermitian matrices with real eigenvalues, as can be seen from the matrix

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}. \tag{1.10}$$

Taking $P = 4$ and $R = 2$ here, we would get $G = (2, 4)$, whereas $\lambda_1 = \lambda_2 = 3$.

DISTRIBUTION THEOREM. *The numbers of eigenvalues of the matrix H of the Separation Theorem which lie to the left and right outside the interval G are equal to the orders of P and R, respectively.*

Proof. Consider the matrix

$$H(\varepsilon) = \begin{bmatrix} P & \varepsilon Q \\ \varepsilon Q^* & R \end{bmatrix} \tag{1.11}$$

for values of ε satisfying $0 \leqslant \varepsilon \leqslant 1$. The matrix $H(0)$ will have the same eigenvalues as the submatrices P and R, so that s and t of them lie to the right and left of the interval G, respectively. When ε is increased continuously from $\varepsilon = 0$ to $\varepsilon = 1$, the eigenvalues will change continuously too, since, as shown by Ostrowski [2, p. 282], the eigenvalues are continuous functions of the elements of the matrix. From the Separation Theorem it follows that during this continuous change from $H(0)$ to $H(1)$ no eigenvalue can lie inside G; hence the number of eigenvalues which lie right and left outside the interval G during this change are s and t, respectively. As $H = H(1)$, the theorem is proved.

Applying this result to the matrix of (1.9) we find that $\lambda_1 \geqslant 2$, $\lambda_2 \leqslant -2$ and $\lambda_3 \leqslant -2$.

Further details about the change of the eigenvalues when ε is increased continuously are provided by the next theorem, which will show that this change is monotonous.

MONOTONY THEOREM 1. *If the Hermitian $n \times n$ matrices*

$$H' = \begin{bmatrix} P & \rho Q \\ \rho Q^* & R \end{bmatrix} \quad \textit{and} \quad H'' = \begin{bmatrix} P & \rho' Q \\ \rho' Q^* & R \end{bmatrix} \tag{1.12}$$

are partitioned such that P is an $s \times s$ and R a $t \times t$ submatrix, and all *the eigenvalues of P are greater than all the eigenvalues of R, then for each pair of real numbers (ρ, ρ') for which $0 \leqslant \rho' \leqslant \rho$, the eigenvalues λ_k' of H' and the eigenvalues λ_k'' of H'' satisfy the inequalities*

$$\lambda_i'' \leqslant \lambda_i' \qquad (1 \leqslant i \leqslant s), \tag{1.13}$$

$$\lambda_j'' \geqslant \lambda_j' \qquad (s < j \leqslant n). \tag{1.14}$$

Proof. Since the eigenvalues of H'' are continuous functions of ρ', as already discussed in the proof of the Distribution Theorem, we may restrict ourselves to $0 < \rho' < \rho$ instead of $0 \leqslant \rho' \leqslant \rho$. Furthermore we may assume, without loss of generality, that $\rho' = 1$ since we may substitute $\hat{Q}$ for $\rho' Q$; the matrix ρQ is then equal to $\hat{\rho}\hat{Q}$ with $\hat{\rho} = \rho/\rho' > 1$. We also make the assumption, again without loss of generality, that the matrix R has nonpositive eigenvalues since this can be attained by adding to H' and H'', without affecting the interrelationship of their eigenvalues, the matrix $-cI_n$, where $c = \lambda_{\max}^{(R)}$. Hence we may write

$$H' = \begin{bmatrix} P & \rho Q \\ \rho Q^* & R \end{bmatrix} \quad \text{and} \quad H'' = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix} \qquad \text{with} \quad \rho > 1, \tag{1.15}$$

where P has positive and R nonpositive eigenvalues.

We now introduce the matrix

$$H''' = \begin{bmatrix} P & \rho Q \\ \rho Q^* & \rho^2 R \end{bmatrix} = \begin{bmatrix} I_s & 0 \\ 0 & \rho I_t \end{bmatrix} H'' \begin{bmatrix} I_s & 0 \\ 0 & \rho I_t \end{bmatrix}, \tag{1.16}$$

having the eigenvalues ν_k $(1 \leqslant k \leqslant n)$ and the orthonormal eigenvectors U_k $(1 \leqslant k \leqslant n)$, and first of all compare the eigenvalues of this matrix with those of H''.

Using the Courant-Fischer theorem [5, p. 101], we have for any eigenvalue λ_i'' of H'' the equality

$$\lambda_i'' = \min_{R_{n+1-i}} \left[\max_{y \in R_{n+1-i}} \frac{y^* H'' y}{y^* y} \right], \tag{1.17}$$

where R_{n+1-i} is an $(n+1-i)$-dimensional linear subspace of the n-dimensional linear space R_n. In addition we define a special $(n+1-i)$-dimensional subspace:

$$R_{n+1-i}^* = \begin{bmatrix} I_s & 0 \\ 0 & \rho I_t \end{bmatrix} (U_i, U_{i+1}, \ldots, U_n), \tag{1.18}$$

this being the subspace spanned up by the independent vectors

$$\begin{bmatrix} U_k^{(1)} \\ \rho U_k^{(2)} \end{bmatrix} \qquad (i \leqslant k \leqslant n). \tag{1.19}$$

(This partitioning is in accordance with the partitioning of H' and H''.) From (1.17) and (1.18) it follows that

$$\lambda_i'' \leqslant \max_{y \in R^*_{n+1-i}} \frac{y^* H'' y}{y^* y} = \frac{y_0^* H'' y_0}{y_0^* y_0}, \tag{1.20}$$

where

$$y_0 = \begin{bmatrix} I_s & 0 \\ 0 & \rho I_t \end{bmatrix} \sum_{k=i}^{n} \alpha_k U_k. \tag{1.21}$$

Since U_k $(1 \leqslant k \leqslant n)$ is an orthonormal set of eigenvectors of H'', we get from (1.16) and (1.21)

$$\begin{aligned} y_0^* H'' y_0 &= \left(\sum_{k=i}^{n} \bar{\alpha}_k U_k^* \right) H''' \left(\sum_{j=i}^{n} \alpha_j U_j \right) \\ &= \sum_{k=i}^{n} |\alpha_k|^2 \nu_k. \end{aligned} \tag{1.22}$$

Therefore, since $\nu_i \geqslant \nu_{i+1} \geqslant \cdots \geqslant \nu_n$,

$$y_0^* H'' y_0 \leqslant \nu_i \sum_{k=i}^{n} |\alpha_k|^2. \tag{1.23}$$

Further, relation (1.21) and $\rho > 1$ give

$$\begin{aligned} y_0^* y_0 &= \left(\sum_{k=i}^{n} \bar{\alpha}_k U_k^* \right) \begin{bmatrix} I_s & 0 \\ 0 & \rho^2 I_t \end{bmatrix} \left(\sum_{j=i}^{n} \alpha_j U_j \right) \\ &\geqslant \left(\sum_{k=i}^{n} \bar{\alpha}_k U_k^* \right) \left(\sum_{j=i}^{n} \alpha_j U_j \right) = \sum_{k=i}^{n} |\alpha_k|^2. \end{aligned} \tag{1.24}$$

For each index i which satisfies $1 \leqslant i \leqslant s$ we find, on applying the Distribution Theorem to the matrix H''', that, since P has positive eigenvalues,

$$\nu_i \geqslant \lambda_{\min}^{(P)} > 0. \tag{1.25}$$

With (1.20), (1.23), and (1.24) this gives

$$\lambda_i'' \leqslant \frac{y_0{}^*H''y_0}{y_0{}^*y_0} \leqslant \frac{\nu_i \sum_{k=i}^{n} |\alpha_k|^2}{\sum_{k=i}^{n} |\alpha_k|^2} = \nu_i \qquad (1 \leqslant i \leqslant s). \tag{1.26}$$

Next we compare the eigenvalues of H''' with those of H'. Applying the Courant-Fischer theorem [5, p. 102] to the matrix equation

$$H''' = H' + \begin{bmatrix} 0 & 0 \\ 0 & (\rho^2 - 1)R \end{bmatrix}, \tag{1.27}$$

we obtain

$$\nu_k \leqslant \lambda_k' \qquad (1 \leqslant k \leqslant n), \tag{1.28}$$

since the last matrix in (1.27) has nonpositive eigenvalues. From (1.26) and (1.28) we finally get

$$\lambda_i'' \leqslant \lambda_i' \qquad (1 \leqslant i \leqslant s). \tag{1.29}$$

The second inequality (1.14) can be proved in a similar way by considering $-H'$ and $-H''$ instead of H' and H''.

Before this theorem can be extended to complex numbers ρ it must be recalled that for each complex number α the matrices

$$H(\alpha) = \begin{bmatrix} P & \alpha Q \\ \bar{\alpha} Q^* & R \end{bmatrix} \tag{1.30}$$

and

$$H(|\alpha|) = \begin{bmatrix} P & |\alpha| Q \\ |\alpha| Q^* & R \end{bmatrix} \tag{1.31}$$

have equal eigenvalues. This can easily be verified as follows. If $\alpha = |\alpha| e^{i\varphi}$, then

$$H(\alpha) = PH(|\alpha|)P^{-1} \tag{1.32}$$

with

$$P = \begin{bmatrix} e^{i\varphi} I_s & 0 \\ 0 & I_t \end{bmatrix} \tag{1.33}$$

and $H(|\alpha|)$ and $PH(|\alpha|)P^{-1}$ have the same eigenvalues. Hence we arrive at the following theorem.

MONOTONY THEOREM 2. *For each pair of complex numbers* (ρ', ρ) *for which* $0 \leqslant |\rho'| \leqslant |\rho|$, *the eigenvalues* λ_k'' *of the Hermitian* $n \times n$ *matrix*

$$H'' = \begin{bmatrix} P & \rho' Q \\ \bar{\rho}' Q^* & R \end{bmatrix} \tag{1.34}$$

and the eigenvalues λ_k' *of the Hermitian* $n \times n$ *matrix*

$$H' = \begin{bmatrix} P & \rho Q \\ \bar{\rho} Q^* & R \end{bmatrix} \tag{1.35}$$

satisfy the inequalities

$$\lambda_i'' \leqslant \lambda_i' \qquad (1 \leqslant i \leqslant s), \tag{1.36}$$

$$\lambda_j'' \geqslant \lambda_j' \qquad (s < j \leqslant n), \tag{1.37}$$

provided that all the eigenvalues of the $s \times s$ *matrix* P *are greater than all the eigenvalues of the Hermitian* $t \times t$ *matrix* R.

A special case of the Monotony Theorem with $\rho' = 0$ and $\rho = 1$ gives rise to

COROLLARY 1. *The eigenvalues of the matrix* H *of Theorem* 1 *satisfy the inequalities*

$$\lambda_i^{(H)} \geqslant \lambda_i^{(P)} \qquad (1 \leqslant i \leqslant s), \tag{1.38}$$

$$\lambda_{j+s}^{(H)} \leqslant \lambda_j^{(R)} \qquad (1 \leqslant j \leqslant t). \tag{1.39}$$

2. APPLICATIONS

When it comes to practical applications, it is sometimes rather difficult to ascertain whether there are submatrices P and R such that the eigenvalues of P and the eigenvalues of R lie separated. However, a criterion which gives always sufficient conditions is Gerschgorin's theorem [**5**, p. 71]. From this theorem it follows that the eigenvalues of the $s \times s$ submatrix P and those of the $t \times t$ submatrix R lie separated if

$$p_{ii} - \Lambda_i^{(P)} > r_{jj} + \Lambda_j^{(R)} \tag{2.1}$$

for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, t$, where $\Lambda_i^{(P)}$ and $\Lambda_j^{(R)}$ are defined by

$$\Lambda_i^{(P)} = \sum_{l=1,\, l\neq i}^{s} |p_{il}|, \qquad \Lambda_j^{(R)} = \sum_{l=1,\, l\neq i}^{t} |r_{jl}|. \tag{2.2}$$

The interval $G' = \{x|\max(r_{jj} + \Lambda_j) < x < \min(p_{ii} - \Lambda_i)\}$ is a subinterval of the interval G of the Separation Theorem.

Two applications of the foregoing theorems will now be discussed. The first concerns a Hermitian matrix of the form

$$H = D + E, \tag{2.3}$$

where the elements e_{ij} of E satisfy

$$|e_{ij}| \leqslant \varepsilon \qquad (1 \leqslant i, j \leqslant n), \tag{2.4}$$

while D is $\operatorname{diag}(d_{kk})$. This kind of matrix is often encountered in the numerical analysis of matrix problems. It is known [5, p. 103] that the eigenvalue $\lambda_l^{(H)}$ satisfies

$$d_{ll} - n\varepsilon \leqslant \lambda_l^{(H)} \leqslant d_{ll} + n\varepsilon. \tag{2.5}$$

Assuming that the diagonal elements d_{kk} are arranged in nondecreasing order (which can always be ensured by a suitable permutation of rows and columns), we conclude from (2.5) that the eigenvalue $\lambda_l^{(H)}$ is distinct from the other eigenvalues of H if

$$d_{l+1,l+1} + 2n\varepsilon < d_{ll} < d_{l-1,l-1} - 2n\varepsilon. \tag{2.6}$$

However, after partitioning as in (1.3) with $s = l$, we find on using Gerschgorin's theorem for the submatrix P,

$$\lambda_{\min}^{(P)} \geqslant \min_{1\leqslant i\leqslant l} [h_{ii} - \Lambda_i].$$

Since $h_{ii} - \Lambda_i \geqslant d_{ii} - \varepsilon - (l-1)\varepsilon = d_{ii} - l\varepsilon$, we get

$$\lambda_{\min}^{(P)} \geqslant d_{ll} - l\varepsilon, \tag{2.7}$$

and on applying the same theorem to the submatrix R we get, along the same lines,

$$\lambda_{\max}^{(R)} \leqslant d_{l+1,l+1} + (n-l)\varepsilon. \tag{2.8}$$

From (2.7) and (2.8) it follows that the condition

$$\lambda_{\min}^{(P)} > \lambda_{\max}^{(R)} \tag{2.9}$$

is satisfied for $s = l$ if

$$d_{ll} > d_{l+1,l+1} + n\varepsilon. \tag{2.10}$$

We then obtain, using Corollary 1 and (2.7),

$$\lambda_l^{(H)} \geqslant \lambda_{\min}^{(P)} \geqslant d_{ll} - l\varepsilon. \tag{2.11}$$

After partitioning H for $s = l - 1$, we find along the same lines that condition (2.9) is satisfied for $s = l - 1$ if

$$d_{ll} < d_{l-1,l-1} - n\varepsilon. \tag{2.12}$$

Again using Corollary 1, we get

$$\lambda_l^{(H)} \leqslant \lambda_{\max}^{(R)} \leqslant d_{ll} + (n - l + 1)\varepsilon. \tag{2.13}$$

Thus we have found that $\lambda_l^{(H)}$ is distinct from the other eigenvalues of H if the condition

$$d_{l+1,l+1} + n\varepsilon < d_{ll} < d_{l-1,l-1} - n\varepsilon, \tag{2.14}$$

which is weaker than (2.6), is satisfied. In addition we get instead of (2.5) the sharper estimate

$$d_{ll} - l\varepsilon \leqslant \lambda_l^{(H)} \leqslant d_{ll} + (n - l + 1)\varepsilon. \tag{2.15}$$

In the second application we examine the eigenvalues of a Hermitian $n \times n$ matrix $A + iB$ (A and B real) in relation to the eigenvalues of the real symmetric matrix A. It is well known [4] that, if λ is an eigenvalue of $A + iB$, the real symmetric matrix

$$H = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \tag{2.16}$$

has a double eigenvalue λ. Defining the number Δ as

$$\Delta = \lambda_{\max}^{(A)} - \lambda_{\min}^{(A)} > 0, \tag{2.17}$$

let us now consider the matrix

$$C = \begin{bmatrix} A + \Delta I_n & -B \\ B & A - \varepsilon I_n \end{bmatrix} \tag{2.18}$$

with $\varepsilon > 0$. Since C is Hermitian and all the eigenvalues of $A + \Delta I_n$ are greater than those of $A - \varepsilon I_n$, it follows from the Distribution Theorem (with $P = A + \Delta I_n$, $R = A - \varepsilon I$, and $Q = -B$) for the eigenvalues of C that

$$\lambda_i^{(C)} \geqslant \lambda_{\min}^{(A)} + \Delta = \lambda_{\max}^{(A)} \qquad (1 \leqslant i \leqslant n), \tag{2.19}$$

$$\lambda_j^{(C)} \leqslant \lambda_{\max}^{(A)} - \varepsilon \qquad (n < j \leqslant 2n). \tag{2.20}$$

Applying the Courant-Fischer theorem to the matrix equation

$$H = C + \begin{bmatrix} -\Delta I_n & 0 \\ 0 & \varepsilon I_n \end{bmatrix}, \tag{2.21}$$

we get [5, p. 102]

$$-\Delta + \lambda_k^{(C)} \leqslant \lambda_k^{(H)} \leqslant \lambda_k^{(C)} + \varepsilon \qquad (1 \leqslant k \leqslant 2n). \tag{2.22}$$

Equations (2.19), (2.20), and (2.22) give

$$\lambda_i^{(H)} \geqslant \lambda_i^{(C)} - \Delta \geqslant \lambda_{\max}^{(A)} - \Delta = \lambda_{\min}^{(A)} \qquad (1 \leqslant i \leqslant n), \tag{2.23}$$

$$\lambda_j^{(H)} \leqslant \lambda_j^{(C)} + \varepsilon \leqslant \lambda_{\max}^{(A)} \qquad (n < j \leqslant 2n), \tag{2.24}$$

from which we conclude that, if the number of eigenvalues of H which are smaller than $\lambda_{\min}^{(A)}$, which lie in the closed interval $[\lambda_{\min}^{(A)}, \lambda_{\max}^{(A)}]$, and which are greater than $\lambda_{\max}^{(A)}$ are p, q and r, respectively, then

$$q + r = n \quad \text{and} \quad p + q = n. \tag{2.25}$$

Hence $p = r$; this means that the number of eigenvalues of H smaller than $\lambda_{\min}^{(A)}$ is equal to the number of eigenvalues of H which are greater than $\lambda_{\max}^{(A)}$. Since, as already pointed out, the eigenvalues of $A + iB$ are double eigenvalues of H we find that:

The number of eigenvalues of the Hermitian matrix $A + iB$ which lie to the left and right outside the closed interval $[\lambda_{\min}^{(A)}, \lambda_{\max}^{(A)}]$ are equal.

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